# CAPITAL ALLOCATION

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July 22, 2007

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## Abstract

This paper investigates procedures for allocating capital in banks and other financial institutions. Capital should be allocated in proportion to the marginal default value of each line of business, where marginal default value is the derivative of the value of the bank's default put with respect to a change in the scale of the business. Marginal default values give a unique allocation that adds up exactly. Marginal default values depend in part on capital allocations. Cross subsidies are avoided if capital allocations are set so that capital-adjusted marginal default values are the same for all lines. Each line's capital allocation should depend on the value of the line's payoffs in default. We include a series of examples showing how our procedures work and we sketch how the procedures would be applied in practice.

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# CAPITAL ALLOCATION

## 1. Introduction

This paper investigates procedures for allocating capital in financial institutions. We focus on banks, but the procedures apply generally.

Consider a bank that diversifies across different activities and asset classes, which may include lending; trading and market making; investment banking; asset management, and retail services, such as credit- and debit-card operations. Financing comes from deposits and debt and from risk capital, which is primarily common equity. The capital has to be sufficient to satisfy regulators and lenders. That is, the bank has to carry enough capital to keep the probability of default or financial distress to an acceptably tiny level. The amount of capital required depends on the risk of the bank's various lines of business. Some businesses are riskier than others, and risky businesses require more capital backing than safe ones.

If the bank can identify capital requirements by lines of business, then it can allocate capital back to the businesses. Allocation is important if capital is costly, say because of tax, information problems or agency costs, or if capital is limited. Limited capital has a shadow price if the firm is forced to pass up positive-NPV investments.

If capital is costly, capital allocation (explicit or implicit) is required to assess the profitability of each line of business and to set incentives and compensation correctly. Allocation is relevant for pricing products and services. The more capital a product or service requires, the higher the break-even price. Allocation is also necessary to calculate the net benefits of hedging or securitization. For example, suppose that a credit-derivatives strategy can offset one half of the risk of a loan portfolio, freeing up half of

the capital that the portfolio would otherwise require. The bank must then compare the costs of hedging to the value of the capital released. To do that, the bank has to know how much capital was absorbed by the loan portfolio in the first place.

If the bank had only one line of business, its capital requirement could be based on stand-alone risk -- on its value-at-risk (VAR), for example. Capital allocations are implicit in RAROC (risk-adjusted return on capital) calculations, where lines of business are assessed capital charges proportional to their VARs. But VARs for several lines of business don't add up. Thanks to diversification, the VAR for the bank as a whole is less than the weighted sum of the stand-alone VARs of the individual businesses. The proper procedures for allocating this diversification benefit are not obvious, and it appears that varying procedures are used in practice.[1] Moreover, there is influential opinion that the bank should not even attempt a complete allocation of capital:[2]

> [T]he risk capital of a multi-business firm is less than the aggregate risk capital of the businesses on a stand-alone basis. Full allocation of risk capital across the individual businesses of the firm is generally not feasible. Attempts at such a full allocation can significantly distort the true profitability of individual businesses.

It's true that the reduction in risk from diversification across lines of business cannot be uniquely allocated to the lines. But we show that the lines' marginal capital requirements are unique and do add up exactly. We argue that capital allocations should be based on these marginal requirements, both in principle and in practice.

Our conclusion that marginal capital requirements add up is not wholly original. One special case has been noted already: *Contribution* VARs, which depend on the

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[1] See Helbekkmo (2006), for example.
[2] Merton and Perold (1993), p. 30.

covariances or betas of the line-by-line returns vs. returns for the bank as a whole, do add up. See Saita (1999) and Stulz (2003),[3] for example. This result falls out from mean-variance portfolio theory. For example, the variance of a bank's overall return -- the return on its portfolio of businesses -- can be expressed as a weighted sum of covariances of each business's return with the overall portfolio return. But it appears that contribution VARs are rarely used for capital allocation. Also, contribution VARs only work in a mean-variance setting. We derive a general adding-up theorem, which works for any joint probability distribution of returns. Our only assumption is complete markets, complete enough that individual lines of business would have well-defined market values if they could be traded separately.

This paper extends Myers and Read (2001), who analyze capital (surplus) allocation for insurance companies.[4] Principles are similar here, although this application is different and more general. For example, insurance risks come from the policies issued on the liability side of the balance sheet. In banks, most of the action is on the asset side of the balance sheet.

The academic and applied literature on VAR and risk management is enormous. Prior work on capital allocation seems much more limited, however. We have quoted Merton and Perold (1993), who question whether capital can or should be allocated back to lines of business. We focus on marginal changes in individual businesses, holding

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[3] Contribution VARs appear in Froot and Stein (1998, pp. 67-68), Stoughton and Zechner (2007), Saita (1999), Stulz (2003, pp. 99-103), and no doubt in other places. The label varies: synonyms for "contribution" include "marginal," for example in Saita and Stulz. Others refer to "incremental VAR," which is not the same thing. Incremental VAR is the discrete change in VAR from adding or subtracting an asset or business from the bank's overall portfolio. Merton and Perold (1993), Perold (2005) and Turnbull (2000) focus on incremental VAR.

[4] Follow-on articles in the insurance literature include Cummins, Lin, and Phillips (2006) and Grundl and Schmeiser (2007).

constant the list of businesses in the bank's portfolio. They focus on investment, that is, decisions to add or subtract an entire line of business. See also Perold (2005). We agree with these authors' starting point, however. They define "risk capital" as the present-value cost of acquiring complete credit protection for the bank. We start with the present value of the bank's default put. The two present values should be the same.

Froot and Stein (1998) consider capital allocation, but their main interest is how banks invest capital, not how to allocate an existing stock of capital to a portfolio of existing businesses. They show that value-maximizing banks will act as if risk-averse, even in perfect financial markets, if investment opportunities are uncertain and raising equity capital on short notice is costly. They discuss contribution VAR and the problems of implementing RAROC. They do not consider default, however. Turnbull (2000) extends this line of research, introducing default risk. Stoughton and Zechner (2007) add a focus on information and agency costs internal to the firm.

Section 2 of this paper presents our adding-up theorem and our approach to capital allocation. We show that capital allocation for a line of business should depend on the business's marginal contribution to default value, defined as the present value of the bank's default put option. We focus on present values, not on VARs or on probabilities of default or financial distress. Section 3 covers specific cases. We present a series of numerical examples based on lognormal returns to show how capital allocation works. The examples indicate that marginal capital allocations are robust to changes in the composition of business; when a business is phased in or out of a bank's portfolio, capital allocations for existing or remaining businesses are reasonably stable. At the end of

section 3 we present two examples of capital allocations with other probability distributions.

Suppose capital is allocated according to our procedures. What is next? How should the allocations affect pricing, performance measurement and investment? Answers to these questions must depend on why bank capital is costly. We review possible costs and their implications in Section 4 and conclude in Section 5.

## 2. Default Values and Capital Allocation

Consider a bank with the following market-value balance sheet:

| Assets | Liabilities |
|---|---|
| Assets $(A = A_1 \dots A_M)$ | Debt (D) |
| Default Put (P) | Equity (E) |
| Franchise value (G) | |

The bank's existing lines of business $A_1$ to $A_M$ are assumed marked to market. The bank's "franchise value," which includes intangible assets and the present value of future growth opportunities, is entered as G. We assume for simplicity that G disappears if the bank defaults in the current period. That is, $G = 0$ in bankruptcy.

The *default-risk free* value of debt and deposits is D. Default risk is captured not in the stated value of debt and deposits, but on the other side of the balance sheet as the default value P, the present value of the bank's default put over the next period. (A period could be a month, quarter or year, but probably not longer.)

If the bank defaults, the default-put value equals the shortfall of end-of-period asset value from end-of-period debt and deposits, including interest due. Lenders or depositors do not necessarily bear this shortfall. The put payoff will be covered, at least

6

in part, by deposit guarantees or other credit backup from the government. Who bears losses in default does not matter for our analysis, however. We do not need to model deposit insurance explicitly. We do assume that any costs of the insurance or other forms of credit backup are sunk and already paid for.

Equity (E) is the market value of the bank's equity, common stock plus issues of preferred stock or subordinated debt that count as capital. The bank's capital C is not the same thing as its equity, however. The capital-account balance sheet is:

| Assets | Liabilities |
|---|---|
| Assets (A) | Debt (D) |
| | Capital (C) |

Capital is C = A - D, the difference between the market value of the bank's assets and the default-risk free value of its debt and deposits. In practice, some of the bank's assets may not be marked to market. The important distinction here, however, is that capital does not reflect default value P or include the intangible assets or future growth opportunities in G.

We assume the asset portfolio consists of M assets (lines of business) with start-of-period values $A_i$. Thus $A = \sum_{i=1}^{M} A_i$. The value of the default put is:

$$P = PV[\max\{0, (R_D D - R_A A)\}], \tag{1}$$

where $R_D$ is the gross payoff to a dollar of debt or deposits (one plus a safe interest rate) and $R_A$ is the uncertain gross return on the bank's assets. All returns are assumed to be uncertain except for $R_D$.[5]

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[5] We take D, the face amount of debt and deposits, as fixed. $R_D D$ is the exercise price of the default put. We could allow for uncertain liabilities, for example insurance contracts, as in Myers and Read (2001). But in

7

Define the *marginal default value* of asset i as $p_i = \partial P / \partial A_i$, the partial derivative of overall put value P with respect to $A_i$. We can show that these marginal default-option values add up uniquely. The sum of the products of each asset and its marginal default value equals the default value of the bank as a whole.

**Theorem 1** The default value P can be allocated uniquely across assets, proportional to the assets' marginal default values $p_i$:

$$P = \sum_{i=1}^{M} p_i A_i. \tag{2}$$

A proof of the theorem for the two-asset case follows. Generalization to M assets is easy. The two assets' uncertain end-of-period payoffs are $R_1$ and $R_2$. The portfolio payoff is:

$$R_A = \frac{R_1 A_1 + R_2 A_2}{A}.$$

The end-of-period promised payoff to debt and deposits, including interest, is $R_D D$. With complete markets, the present value of the default put is:

$$P = \int_Z \pi(z)[R_D D - R_1 A_1 - R_2 A_2] dz, \tag{3}$$

where $\pi(z)$ is a state-price density for each combination of $R_1$ and $R_2$ in the region Z where the put is in the money. Figure 1 plots this region. Each state z is a unique point in the in-the-money region Z. Each point is a combination of returns on the two assets ($R_1$,

this context it's easier to think of a risky liability as a short position in a risky asset. The next section includes an example of capital allocation to a short position.

$R_2$), which generates a portfolio return of $R_A = R_1 A_1 + R_2 A_2$. The valuation equation (2) sums across the continuum of states, with the payoff in each state z multiplied by the state-price density $\pi(z)$. Note that the states are identified by the portfolio return $R_A$ and that the state prices $\pi(z)$ are fixed. Therefore an extra dollar delivered in state z by asset $A_1$ has exactly the same present value as an extra dollar delivered by $A_2$. The valuation formula sums across states.

The amount of debt and deposits depends on A and a parameter c, the *capital ratio*, which measures the amount of capital that the bank puts up to back its liabilities. The capital ratio is a choice made by the bank or its regulators. For now we take c as constant across the bank's lines of business, with $D = (1 - c)(A_1 + A_2)$. Therefore:

$$
\begin{aligned}
P &= \int_Z \pi(z)[R_D(1-c)(A_1 + A_2) - R_1 A_1 - R_2 A_2]dz \\
&= A_1 \int_Z \pi(z)[R_D(1-c) - R_1]dz + A_2 \int_Z \pi(z)[R_D(1-c) - R_2]dz.
\end{aligned}
\tag{4}
$$

Changes in $A_1$ and $A_2$ affect limits of integration at the boundary of region Z in Figure 1. These marginal effects can be left out, however, because the put payoff on the boundary is zero.[6] Thus $p_1$ and $p_2$ are:

$$
p_1 = \frac{\partial P}{\partial A_1} = \int_Z \pi(z)[R_D(1-c) - R_1]dz,
$$

$$
p_2 = \frac{\partial P}{\partial A_2} = \int_Z \pi(z)[R_D(1-c) - R_2]dz.
\tag{5}
$$

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[6] Even if there were value effects from shifts of the boundary, we can show that the effects would cancel. Appendix 1 in Myers and Read (2001) shows how boundary changes cancel.

Multiply $p_1$ and $p_2$ by the respective asset values $A_1$ and $A_2$ to get the adding-up result, $p_1 A_1 + p_2 A_2 = P$.

It's clear from the valuation formulas that an across-the board expansion of assets and liabilities (with c constant) will result in a proportional increase in overall default value. Given c, $p \equiv \partial P / \partial A$ is a constant for any proportional change, regardless of the size of the change.

Expansion of a single line of business will also affect P, but not proportionally. If c is constant, marginal default values $p_i$ will vary across lines of business, and can be negative for relatively safe assets. For example, for a risk-free asset, where $R_i = R_D$,

$$p_i = \int_Z \pi(z)[-cR_D]dz < 0.$$

A bank that allocates capital proportional to assets, despite varying marginal default values, is forcing some businesses to cross-subsidize others. This contaminates performance measurement, incentives and compensation, pricing, and decisions about securitization and hedging. The remedy is to vary capital allocation inversely to marginal default values, so that each business's capital-adjusted contribution to default value is the same.

Write out the value of the default put allowing variation in marginal capital ratios $c_i$:

$$P = \int_Z \pi(z)[\sum A_i (1-c_i)R_D - \sum R_i A_i]dz$$

$$p_i = \int_Z \pi(z)[(1-c_i)R_D - R_i]dz. \tag{6}$$

Our adding-up theorem still holds. Also, an increase in the marginal capital allocation $c_i$ always decreases the exercise price of bank's default put and reduces its value. Therefore, we can offset differences in $p_i$ with compensating variation in the capital allocations $c_i$. Examples are given in the next section of this paper. The following is a more general statement of what the allocations depend on.

The valuation expressions can be simplified by defining

$$\Pi_Z(R_X) \equiv \int_Z \pi(z) R_X dz. \tag{7}$$

For example, $\Pi_z(R_x)$ is the present value of a safe asset's return (but only in the in-the-money region Z, like the payoff on a cash-or-nothing put triggered by default). Write marginal default value $p_i$ as:

$$p_i = (1 - c_i)\Pi_Z(R_D) - \Pi_Z(R_i). \tag{8}$$

The overall default value is

$$p = (1 - c)\Pi_Z(R_D) - \Pi_Z(R_A). \tag{9}$$

Solve for the marginal capital allocations that set $p_i = p$:

$$c_i = c + \frac{\Pi_Z(R_A) - \Pi_Z(R_i)}{\Pi_Z(R_D)}. \tag{10}$$

Thus marginal capital allocations for an asset or business should depend on the present value of its returns in default -- that is, on the present value of its returns as distributed across the in-the-money region Z in Figure 1. If its returns are "riskier" than

the overall portfolio return $R_A$ in region Z -- that is, worth less than the overall return in that region -- then $c_i > c$. If its returns in Z are relatively safe and worth more than the overall return, then $c < c_i$. The capital ratio for line i does *not* depend on the line's marginal affect on the probability of default. It depends on the *value* of the line's payoff in default.

We have assumed complete markets, complete enough that the bank's assets and default put option would have well-defined market values if traded separately. Given that assumption, we have shown that capital can be allocated in proportion to the marginal default value of each line of business, where marginal default value is the derivative of the value of the bank's default put with respect to a change in the scale of the business. Marginal default values give a unique allocation that adds up exactly. Differences in marginal default values can be offset by differences in marginal capital allocations. Cross subsidies are avoided if capital ratios are set so that capital-adjusted marginal default values are the same for all lines. Each line's capital ratio should depend on the value of the line's payoffs in default.

## 3. Special Cases and Examples

We now turn to specific cases, starting with the "Black-Scholes" case where the bank's overall return is assumed lognormal. This allows closed-form formulas for marginal default values and capital allocations. We present several numerical examples showing

how our capital allocations depend on risk and on return correlations across different assets or lines of business.[7]

If asset returns are log-normal, the default-put value depends only on $D$, $A$ and $\sigma_A$, where $\sigma_A$ is the volatility of the bank's overall asset portfolio. Thus $P = f(D, A, \sigma_A)$. Since $D = (1 - c)A$, the ratio of put-option value to asset value can be written as a function of capital ratio $c$ and the asset volatility :

$$p = \frac{P}{A} = f((1-c), 1, \sigma_A) = f(c, \sigma_A). \tag{11}$$

(Here it's convenient to define default-option value as a fraction of asset value, that is, as $p = P/A$ rather than $P$, and also convenient to consider changes in the fractional value of each asset $a_i$, where $a_i = A_i/A$.) Taking the derivative of the default value $p$ with respect to $a_i$ gives:

$$p_i = \frac{\partial P}{\partial A_i} = p + \frac{\partial p}{\partial a_i}(1 - a_i). \tag{12}$$

The first term $p$ is the change in default value due to an increase in $A$, the overall value of the bank's assets, ignoring any change in the composition of its assets. The second term captures the change in $p$ due to a change in the composition of the asset portfolio $\partial p/\partial a_i$.

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[7] Assuming lognormal returns is awkward in one respect, because the sum of lognormal variables is not itself lognormal. The following examples assume that the return on the bank's overall portfolio of assets is lognormal. We calculate the volatility of this overall return from the standard deviations of and correlations among the individual assets. At the end of this section we present a different example where returns on individual assets are assumed lognormal.

Equation (9) can also be written as:

$$\frac{\partial p}{\partial a_i} = \frac{\partial p}{\partial c}\frac{\partial c}{\partial a_i} + \frac{\partial p}{\partial \sigma_A}\frac{\partial \sigma_A}{\partial a_i}.$$

Given that[8]

$$\frac{\partial c}{\partial a_i} = \frac{c_i - c}{(1 - a_i)} \text{ and } \frac{\partial \sigma_A}{\partial a_i} = \frac{\sigma_{iA} - \sigma_A^2}{\sigma_A(1 - a_i)},$$

the marginal default-option value for each asset risk type is:

$$p_i = p + \frac{\partial p}{\partial c}\frac{(c_i - c)}{(1 - a_i)} + \frac{\partial p}{\partial \sigma_A}\frac{(\sigma_{iA} - \sigma_A^2)}{\sigma_A}, \tag{13}$$

where $\sigma_{iA}$ is the covariance of the log of the return $R_i$ with the log of the portfolio return $R_A$. Since the bank's future return is lognormal,

$$p = (1 - c)N\{x\} - N\{x - \sigma_A\},$$

$$\frac{\partial p}{\partial c} = -N\{x\} \text{ and } \frac{\partial p}{\partial \sigma_A} = N'\{x - \sigma_A\},$$

where $x = \dfrac{\ln(1 - c)}{\sigma_A} + \dfrac{\sigma_A}{2}.$

The option delta ($\partial p/\partial c$) is negative, so the higher the marginal capital allocation, the higher the ratio of asset value to debt and deposits, and the lower the marginal default

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[8] The last term of the following equation uses

$$\sigma_A^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} a_i a_j \rho_{ij}\sigma_i\sigma_j \text{ and } \sigma_{iA} = a_i\rho_{iA}\sigma_i\sigma_A.$$

value. The option vega ($\partial p / \partial \sigma_A$) is positive, so the higher the covariance of the asset return with the asset portfolio, the higher the marginal default value.

Equation (13) says that assets will have different marginal default values if marginal capital allocations are the same. Cross-subsidies and distortions are avoided only if capital is allocated to equalize marginal default values. Therefore we set $p_i = p$ and solve for $c_i$:

$$c_i = c - \left(\frac{\partial p}{\partial c}\right)^{-1}\left[\frac{\partial p}{\partial \sigma_A}\frac{(\sigma_{iA} - \sigma_A^2)}{\sigma_A}\right].\qquad(14)$$

Thus the marginal capital allocations in the lognormal case depend on $\sigma_{iA}$, the covariance of asset i's return with the overall return, relative to the variance of the overall return $\sigma_A^2$. High-risk assets with $\sigma_{iA} > \sigma_A^2$ must be allocated extra capital at $c_i > c$. (Recall that $\partial p / \partial c$ is negative.) Safer assets with $\sigma_{iA} < \sigma_A^2$ require less capital at $c_i < c$. Relatively safe assets can actually have negative capital allocations, as our numerical examples will show.

Equation (14) says that marginal capital allocations should depend on the covariance of each asset's return vs. the bank's overall return $R_A$, the volatility of $R_A$, and also on the delta and vega of the bank's default put option. The optimal marginal allocations are not proportional to that covariance, but to the difference between the covariance and the variance of $R_A$.

## 3.1. Lognormal Examples

Take a bank that has four lines of business with assets $A_1, A_2, A_3$, and $A_4$. Each line's assets are worth the same amount, say \$100. Total capital is \$32, 8% of total assets of \$400. Standard deviations of the asset returns are 3, 5, 7, and 20%. All pairwise correlation coefficients are 0.1.[9] The table below reports these parameters as well as covariances between each asset return and the return on the bank's overall asset portfolio.

|  | Assets by Line of Business $A_i$ | Standard Deviation $\sigma_i$ | Covariance $\sigma_{iA}$ |
|---|---|---|---|
| $A_1$ | \$100 | 3% | 0.00047 |
| $A_2$ | \$100 | 5% | 0.00100 |
| $A_3$ | \$100 | 7% | 0.00172 |
| $A_4$ | \$100 | 20% | 0.01075 |
| Total | \$400 | 5.9% |  |
| Capital | 8% of \$400 = \$32 |  |  |

The delta and vega of the default put option are -0.083 and 0.141, respectively. Put value is \$0.81 (80¢), that is, p = 0.202% of the total asset value. This is a very small number, only 20 basis points. It should be a small number, because prudent management and regulation reduces the odds of default in any period almost to zero. The default value is not economically trivial, however.

Suppose that capital is allocated proportional to assets, 8% of each line's assets. The marginal default values are:[10]

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[9] We set the risk-free rate of return equal to zero ($R_D = 1$) in all numerical examples.
[10] The marginal default values are here given in dollars, the product of $p_i$ and $A_i$. Take asset 1 as an example. The partial derivative of p with respect to $A_1$ is $p_1 = \partial p / \partial A_1$ = - 0.52%. Asset 1's marginal dollar contribution to default value is - 0.52% of \$100 = - \$0.52.

| Assets (Standard Deviations) | Marginal Default Value ($) | Capital Allocation ($) |
|---|---|---|
| $A_1 = \$100$ (3%) | -0.52 | 8 |
| $A_2 = \$100$ (5%) | -0.39 | 8 |
| $A_3 = \$100$ (7%) | -0.22 | 8 |
| $A_4 = \$100$ (20%) | 1.94 | 8 |
| $A = \$400$ (5.9%) | 0.202% of $400 = 0.81 | 32 |

The marginal default values add up but are not the same. Marginal contributions to default value for lines 1, 2 and 3 are negative, offset by a large positive contribution for line 4. The assets in line 4 would get a subsidy from the other lines. The capital allocations should therefore be changed as follows:

| Assets (Standard Deviations) | Marginal Default Value ($) | Capital Allocation ($) | Capital-adjusted Contribution to Default Value ($) |
|---|---|---|---|
| $A_1 = \$100$ (3%) | -0.52 | -0.66 | 0.202 |
| $A_2 = \$100$ (5%) | -0.39 | 0.88 | 0.202 |
| $A_3 = \$100$ (7%) | -0.22 | 2.93 | 0.202 |
| $A_4 = \$100$ (20%) | 1.94 | 28.85 | 0.202 |
| $A = \$400$ (5.9%) | 0.81 | 32 | 0.81 |

These allocations eliminate cross-subsidies. Note that the capital allocations still add up exactly. Note also that the allocations are not proportional to the standard deviations or covariances of the individual lines of business.[11] The allocations are

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[11] In this example the covariances are proportional to the standard deviations, because all pairwise correlations are the same.

therefore not proportional to standard VARs based on standard deviations or contribution VARs based on covariances.

It's worth pausing to show how the capital-adjusted contributions are calculated. Take line 4 as the example. With an 8% capital allocation ($c_4 = c = 0.08$), the capital ratio is calculated from Equation (13) (the term with $c_i - c$ drops out):

$$p_4 = p + \frac{\partial p}{\partial \sigma_A} \frac{\left(\sigma_{4A} - \sigma_A^2\right)}{\sigma_A}$$

The inputs are: $p = .00202$, $\partial p / \partial \sigma = .141$, $\sigma_{4A} = .01075$ and $\sigma_A^2 = .059^2 = .00348$.

$$p_4 = .00202 + .141 \frac{(.01075 - .00348)}{.059} = .0194, \text{ or } 1.94\%$$

We have to allocate extra capital to line 4's assets in order to reduce $p_4$ from 1.94% to $p = 0.202\%$. The formula is:

$$p_4 = p + \frac{\partial p}{\partial c}(c_4 - c) + \frac{\partial p}{\partial \sigma_A}\left(\frac{\sigma_{4A} - \sigma_A^2}{\sigma_A}\right)$$

Set $p_4 = .00202$:

$$.00202 = .00202 - .0833(c_4 - .08) + .141\left(\frac{.01075 - .00348}{.059}\right),$$

Solve for $c_4$, which is .2885, or $28.85 against $A_4 = \$100$.

Now consider variations on our basic example. First suppose that the four business lines are independent firms, each with the same 0.202% default value as the four-line firm. The loss of diversification means capital must increase from $32 to $50.7:

| Stand-alone Assets (Standard Deviations) | Capital Required ($) |
|---|---|
| $A_1 = \$100$  (3%) | 3.25 |
| $A_2 = \$100$  (5%) | 6.48 |
| $A_3 = \$100$  (7%) | 9.87 |
| $A_4 = \$100$  (20%) | 31.1 |
| Total Capital Required | 50.7 |

Loss of diversification costs $50.7 - 32 = \$18.7$. There is no way for the diversified four-line bank (in the first example) to allocate its gain from diversification ($18.7) back to its individual lines of business. Fortunately there is no need to do so. We allocate the capital actually required by the diversified firm, not the capital that would have been required by stand-alone businesses.

The next table shows how capital allocations change in the diversified bank when correlations of lines 1 and 2 with line 4 increase from 0.1 to 0.9. Total capital remains at $32, 8% of total assets.

| Assets (Standard Deviations) | Marginal Default Value ($) | Capital Allocation ($) | Capital-adjusted Contribution to Default Value ($) |
|---|---|---|---|
| $A_1 = \$100$  (3%) | -0.52 | 0.80 | 0.408 |
| $A_2 = \$100$  (5%) | -0.16 | 3.61 | 0.408 |
| $A_3 = \$100$  (7%) | -0.50 | 0.90 | 0.408 |
| $A_4 = \$100$  (20%) | 2.81 | 26.69 | 0.408 |
| $A = \$400$  (5.9%) | 0.408% of $400 = \$1.63 | 32 | 1.63 |

The default value about doubles, to $1.63 from $0.81 in the first example. More capital is allocated to assets 1 and 2, because they now contribute more to overall risk. Less capital is allocated to assets 3 and 4.

The final example (for now) assumes that asset 4 is a short position or a risky liability. (In this setup, it's convenient to treat risky liabilities as short positions in risky assets.) Capital is still equal to $32 and all pairwise correlations are 0.1.

| Assets (Standard Deviations) | Marginal Default Value ($) | Capital Allocation ($) | Capital-adjusted Contribution to Default Value ($) |
|---|---|---|---|
| $A_1 = \$300$   (3%) | -2.40 | -3.90 | 0.427 |
| $A_2 = \$100$   (5%) | -0.79 | -1.25 | 0.427 |
| $A_3 = \$100$   (7%) | 1.65 | 17.25 | 0.427 |
| **$A_4 = - \$100$ (20%)** | 3.25 | 19.90 | 0.427 |
| $A = \$400$   (5.9%) | 0.427% of $400 = 1.71 | 32 | 1.71 |

Note that the short position has a *positive* capital allocation equal to -19.9% of - $100 = $19.90. Default value more than doubles, to $1.71 from $0.81 in the original example. The capital allocation to the short position in asset 4 declines to $19.90, however, because the short position is a partial hedge to the risk of the other three assets.

## 3.2. Robustness of Marginal Default Values

Marginal default values depend on the mix of business lines as well as line-by-line risk. When the mix changes, marginal default values change and so do capital allocations. If the bank in our example sells off one of four lines of business, capital allocations for the remaining three lines have to be recomputed. If it acquires a fifth line, capital allocations

for the initial four lines also change. Are the changes a significant practical problem? They could be if our procedures gave capital allocations for remaining or existing businesses that bounce around significantly in response to routine adjustments in business mix.

We have explored this issue numerically. Consider capital allocations for hypothetical companies with N and N + 1 identical lines of business. Assume that asset lines are uncorrelated and that each has an annual standard deviation of 10%. Uncorrelated assets give the largest diversification gains and should generate the largest changes in allocations as assets are added or subtracted. We hold the default value at 0.202% of assets. Figure 2 shows two cases: a bank that has three lines of business and adds a fourth and a bank that has nine lines and adds a tenth. In both cases the allocation for the new line increases as more of the new line is added to the mix. Allocations for existing lines change only gradually, however, and hardly at all in the ten-line example. Capital allocations for existing lines of business appear to be robust when new lines are added or old lines subtracted.

These experiments suggest a practical answer to the problem of allocating diversification gains or losses when there is a significant discrete change in business mix. Allocations for a business that is added or subtracted can be very sensitive to the magnitude of the change. Allocations to existing businesses can be much more stable, however, and for practical purposes may not have to be adjusted.

Consider a proposal to add an entirely new business. The new business's NPV is reduced by the cost of the capital allocated to it. The investment is worthwhile if *net* NPV is positive, taking the mix of existing businesses as constant. Net NPV means Adjusted

Present Value (APV), [12] here equal to NPV plus the cost of allocated capital. Figure 2 suggests that the amount of capital allocated increases steadily as the scale of the new business increases. Thus capital allocation is a source of decreasing returns to investment. [13]

## 3.3. Alternative Distributions

Bank portfolios include assets with very different return distributions. The return to a trading desk with long and short positions does not have the same type of probability distribution as the return on a loan portfolio, for example. Thus default values and capital allocations in real life depend on a mixture of distributions with different shapes and characteristics. Fortunately, our adding-up theorem and capital allocation procedures work for any probability distributions, although computation will usually require numerical procedures in place of closed-form solutions. Here we give two simple examples.

In previous examples, we assumed that the bank's overall portfolio return is lognormal. We used the standard deviations and correlations for individual assets to calculate the volatility of the bank's overall return. But the sum of lognormals is not itself lognormal. Thus by assuming a lognormal portfolio return, we could not consistently assume lognormal returns on individual assets. On the other hand, if individual asset returns are lognormal, the overall return is not.

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[12] APV is described in Ch. 19 of Brealey, Myers and Allen (2006).
[13] Line-by-line APVs could not be used to construct the optimal overall mix of business, however. The APV of each business would depend on the order in which candidate businesses were evaluated. This problem is highlighted by Merton and Perold (1993) and Perold (2005). We discuss the problem in the next section.

The next example takes individual asset returns as lognormal, with the same standard deviations as our initial example. For simplicity we assume all correlations are zero. Capital is $32 against total assets of $400. We calculate default values and capital allocations, using Monte Carlo simulation of Equations (8), (9) and (10).[14] The results are as follows.

| Assets (Standard Deviations) | Marginal Default Value ($) | Capital Allocation ($) |
| --- | --- | --- |
| $A_1 = \$100$   (3%) | -0.41 | -0.76 |
|  | (-0.47) | (-1.03) |
| $A_2 = \$100$   (5%) | -0.30 | 1.04 |
|  | (-0.38) | (0.26) |
| $A_3 = \$100$   (7%) | -0.16 | 3.51 |
|  | (-0.25) | (2.20) |
| $A_4 = \$100$   (20%) | 1.29 | 28.21 |
|  | (1.68) | (30.56) |
| $A = \$400$ | 0.42 | 32.00 |
|  | (0.59) | (32.00) |

The numbers in parentheses are the default values and capital allocations for our original example, except that all correlations are changed to zero. In other words, the numbers in parentheses show the values and allocations obtained if the *overall* bank return is lognormal. Notice how the bank's default put value declines, from $0.59 to $0.42, when *individual* returns are lognormal. Capital allocations change as well. This experiment suggests that capital allocations in practice will be sensitive to changes in assumptions about return distributions on different types of assets and different lines of business.

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[14] Simulations were run in MATLAB with 1 million random draws for each example.

The final example adds a jump to asset 3 – a rare adverse event such as a liquidity crisis. We also change asset 4's distribution from lognormal to normal. A normal distribution is probably a better fit to a trading portfolio. $R_1$ and $R_2$ are still log normally distributed with parameters $\mu_1, \sigma_1$ and $\mu_2, \sigma_2$.

Define $R_3$ as $R_3 = R_3^1 + \theta \varepsilon_3$, where $R_3^1$ is lognormally distributed with parameters $\mu_3$ and $\sigma_3$. Define $\varepsilon_R$ as a Poisson random variable with parameter $\lambda$. The size of the jump $\theta$ is normally distributed with mean $\mu_S$ and variance $\sigma_S^2$. $R_4$, the gross return on asset 4, is normally distributed with mean $\mu_4 = 1$ and variance $\sigma_4^2$. For simplicity we continue to assume that all correlations are zero. We simulate asset and portfolio returns and calculate default values and capital allocations, again using Equations (8), (9) and (10). [15] The results are as follows. [16]

| Assets (Standard Deviations) | Marginal Default Value ($) | Capital Allocation ($) |
|---|---|---|
| $A_1 = \$100$ (3%) | -0.57 | -1.89 |
| $A_2 = \$100$ (5%) | -0.48 | -0.64 |
| $A_3 = \$100$ (7%) | -0.06 | 4.59 |
| $A_4 = \$100$ (20%) | 1.94 | 29.94 |
| $A = \$400$ | 0.83 | 32.00 |

Overall default value increases to $0.83 and additional capital is allocated to assets 3 and 4.

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[15] The parameters $\lambda$, $\mu_S$ and $\sigma_S^2$ are 0.2, -0.1 and 0.05. Means are $\mu_1 = -\tfrac{1}{2}\sigma_1^2$, $\mu_2 = -\tfrac{1}{2}\sigma_2^2$ and $\mu_3 = -\tfrac{1}{2}\sigma_3^2 - E(\theta \varepsilon_R)$, so that in each case the expected value of the gross return is 1.0. The risk-free interest rate is zero and the gross risk-free return is $R_D = 1.0$.

[16] We find that $\Pi_z(R_D) = 0.0789$, $\Pi_z(R_A) = 0.0706$, $\Pi_z(R_1) = 0.0784$, $\Pi_z(R_2) = 0.0774$, $\Pi_z(R_3) = 0.0732$, and $\Pi_z(R_4) = 0.0532$.

24

This example shows again that marginal default values and capital allocations can be sensitive to the shapes of the distributions of asset returns. We plan to extend these examples to somewhat more realistic cases, in order to understand how capital allocations could vary under different distributional assumptions and more reasonable correlations among returns. We will also explore how regulatory capital requirements, for example the Basel II rules, differ from the capital allocations generated by our procedures.

## 4. Some Implications

Now consider how capital allocation affects management and decision making in a bank. We distinguish three settings. In the first "business as usual" setting, the bank's composition of business is constant. That is, the identity and approximate relative magnitudes of its several lines of business are taken as given. Here capital allocations apply to existing businesses and assets. They are relevant for pricing, performance measurement, incentives and compensation, and trading and hedging decisions.

In the second setting, the bank has to decide whether to add or subtract a line of business or a significant block of assets. The decision hinges on whether the bank is better off with or without the business or assets. Capital allocations "with" are not the same as "without." All capital allocations can change after a discrete investment. But if the investment is small, allocations for existing lines can be held constant. Figure 2 indicates that holding existing allocations constant can be a good approximation if the bank has many existing lines of business and if incremental changes are small. (Call this the "incremental changes" setting.) If the approximation is acceptable, then the focus is only on the capital allocated to the new investment, which increases with the scale of

investment.  Optimal scale (holding existing assets constant) is reached when APV (NPV minus the cost of allocated capital) is zero at the margin.

A bank could not be constructed from scratch by this method. Valuing assets one by one, holding other assets constant, cannot work when default value is important and all assets are in play. But we can consider a third setting where management searches for the optimum *portfolio* of businesses, starting with a menu of candidates. This is a mathematical programming problem. Banks solve this problem implicitly when they set strategy or launch takeovers or major restructurings. A sketch of the programming problem follows.[17]

Suppose capital is available at a tax cost $\tau$ per dollar of capital. The objective is to maximize bank value, net of the tax cost $\tau C$. The decision variables are the amounts invested ($INV_i$) in the menu of assets i = 1 … M and the amount of capital C and debt and deposits D raised from depositors or investors. Financing must cover investment, so $\Sigma INV_i = C + D$. Assume a maximum acceptable default put value, expressed as a fraction of total asset value ($p \leq p(max)$). The default put value depends on the scale and mix of assets, the joint probability distribution of the assets' returns and on the amount of capital C. The constraint $p \leq p(max)$ therefore sets the floor for C. This constraint on C will be binding at the optimum. The tax cost $\tau$ of raising C therefore gets passed back as a shadow price to each line of business i.[18] The adjusted present value of business i is $APV_i$

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[17] For simplicity we assume the optimal portfolio is chosen once or for one period only. We hold franchise value and growth opportunities G constant. Dynamics are more complicated. Froot and Stein (1998) introduce some dynamics of bank capital structure decisions.

[18] This result, that an investment's NPV is adjusted to APV depending on the shadow prices of the constraints affected by the investment, follows Myers and Pogue (1974). But they simplify to a linear program by assuming that the debt supported by each investment is a constant fraction of the amount invested. Debt "supported by" an investment is the complement to the capital "required by" the investment $INV_i$.

= $NPV_i - \tau C_i$ ,   where $NPV_i = A_i - INV_i$ and $C_i$ = the marginal capital allocation (in dollars) to asset i. At the optimum, marginal APV is zero.[19]

Note how $APV_i$ depends on the marginal capital allocations $c_i$. At the optimum portfolio, these allocations must make all marginal default values equal at $p_i = p$. Otherwise the program's solution could not be optimal – portfolio value could be increased by investing less in assets with high $p_i$ and more in assets with low $p_i$.

Thus our procedure of setting marginal capital requirements to equalize capital-adjusted marginal default values follows from optimization of the bank's overall composition of businesses. We conclude that our capital-allocation results are relevant in all three settings – business as usual, incremental changes and overall portfolio design.

We have assumed tax costs of holding capital, but there are other possible costs. If raising equity capital is not feasible, capital can be constrained and rationed. The shadow price of the constraint should be deducted from APV, depending on the capital required at the margin for each investment. If raising equity capital is feasible but incurs transaction costs, the marginal transaction costs should be charged against APV in place of the shadow price on the capital constraint.

Bank capital is also said to be costly because of agency and information costs. See Merton and Perold (1993) and Perold (2005), for example. These costs are less clear. For example, if the bank is not fully transparent, additional capital should add value, not

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[19] In corporate finance, the tax-adjustment term in APV is usually expressed as the tax advantage of debt vs. equity. NPV is calculated at a pre-tax opportunity cost of capital, and the present value of interest tax shields is added. See Brealey, Myers and Allen (2006), Ch. 19. The interest tax shields depend on the amount of debt supported by the investment. In our setting, where the bank is allocating capital, NPV should be calculated at an *after-tax* cost of capital, as if the investment were 100% financed by debt and deposits, and the cost of the capital required to support the investment should be subtracted. These alternatives are of course equivalent, two sides of the same coin.

reduce it. Banks do business with counterparties who depend on the bank's credit. If the number of counterparties is large, the total cost of counterparties' due diligence and continuing credit tracking of the bank can be significant. These costs are passed on to the bank as less favorable terms on the banks' transactions. A bank with more capital, other things equal, imposes lower costs on counterparties and should be more profitable. Thus lack of transparency is an argument for more capital, not less.

We have yet to see a good explanation for agency costs of bank capital. Are they costs of free cash flow, where managers are reluctant to curtail investment and release cash to shareholders? Adding debt in place of equity is regarded as a treatment or cure for this free-cash-flow problem. Lambrecht and Myers (2007) show how debt can discipline management and force them to disinvest. But they also show that too much debt and too little equity can force managers to disinvest inefficiently *early*. There is no reason to believe that more debt and less equity always adds value. There is no reason to believe that more capital in a bank always generates more agency costs. [20]

Are the agency rents extracted by insiders? Myers (2000) and Lambrecht and Myers (2007) show that such rents need not interfere with efficient investment and financing. We believe that the agency costs of bank capital have to be thought through much more carefully.

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[20] Of course there are other reasons why more capital can add value. For example, a cushion of extra capital can protect franchise value and forestall regulatory intervention if the bank suffers temporary losses.

## 5. Conclusions

We argue that capital can and should be allocated in proportion to the marginal default value of each line of business, where marginal default value is the derivative of the value of the bank's default put with respect to a change in the scale of the business. Marginal default values give a unique allocation that adds up exactly. This adding-up result requires complete markets, complete enough that the bank's assets and default put option would have well-defined market values if traded separately. This result does not require any assumption about probability distributions of returns.

Marginal default values will differ if capital is allocated proportional to assets. These differences can be offset by changing marginal capital allocations. The adding-up result still holds. Cross subsidies are avoided if allocations are set so that capital-adjusted marginal default values are the same for all lines of business. We show that each line's capital ratio should depend on the present value of the line's payoffs in default.

We illustrate our allocation procedures with a series of numerical examples. The examples confirm that our allocations add up under different assumptions about return distributions. The examples suggest that realistic allocations will be sensitive to the shapes of return distributions for different assets and businesses. Looking just to VAR or contribution VAR may not be enough to allocate capital properly.

Marginal default values and capital allocations depend on the composition of the bank's overall portfolio. When the composition changes, allocations should in principle change also. But our numerical experiments suggest allocations for existing businesses can be held constant when the bank invests in a new business. The capital allocated to the new business depends on the scale of investment, however.

When a bank selects its overall portfolio of assets and businesses, capital allocations based on marginal default values are endogenous. But capital allocations at the optimal portfolio should still be based on marginal default values.

We plan to expand our numerical experiments. Further work is also needed on the regulatory implications of our capital-allocation results. For example, we have equated default to the exercise of a default put by an insolvent bank with assets worth less than debt and deposits. Bank of New England and Barings defaulted in this fashion, as did many S&Ls in the 1980s. But regulators can sometimes intervene before the bank is terminally insolvent. What is the role of capital allocation in that case?

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**FIGURE 1**: Payoffs to the default put in the two-asset case. $R_1$ and $R_2$ are payoffs to assets 1 and 2. $R_D D$ is the promised payoff to debt and deposits, including interest. The default put pays off in region Z. The put payoff is zero on the downward-sloping diagonal boundary of Z.
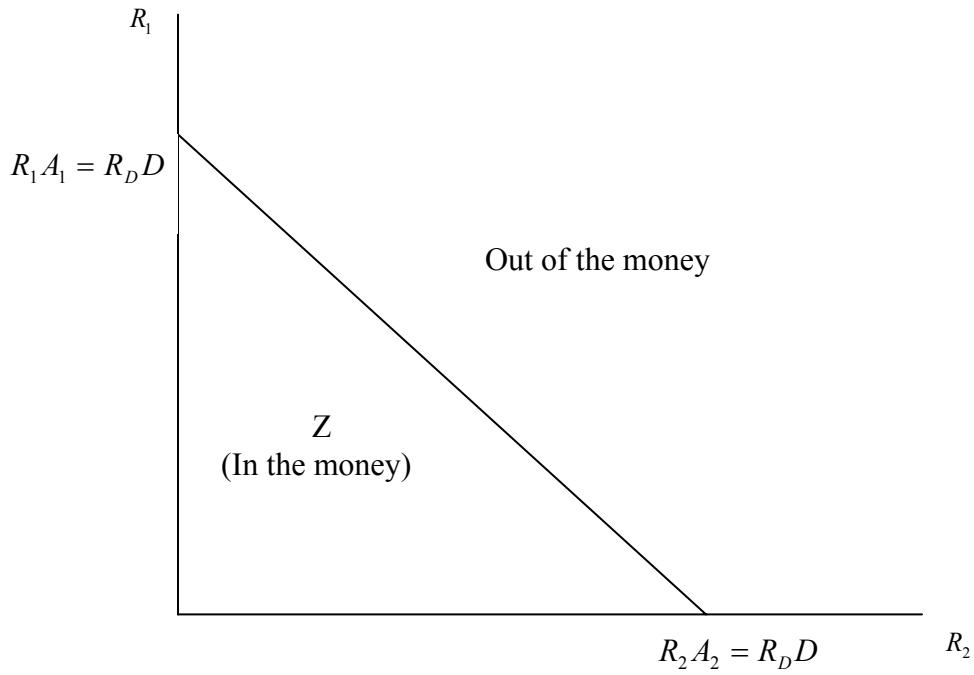
$R_1$

$R_1 A_1 = R_D D$

Out of the money

Z
(In the money)

$R_2 A_2 = R_D D$

$R_2$

**FIGURE 2:** Changes in capital allocations when a new line of business is added to the bank's portfolio.



**4-Line Case**

*Capital Ratio* vs *Percent of Line 4 in the Portfolio*

Legend: Bank, Line 1-3, Line 4



**10-Line Case**

*Capital Ratio* vs *Percent of Line 10 in the Portfolio*

Legend: Bank, Line 1-9, Line 10